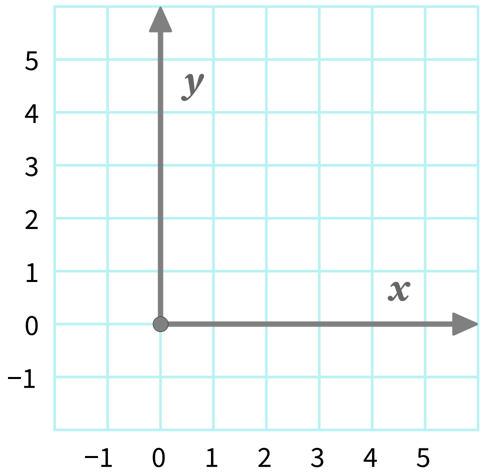
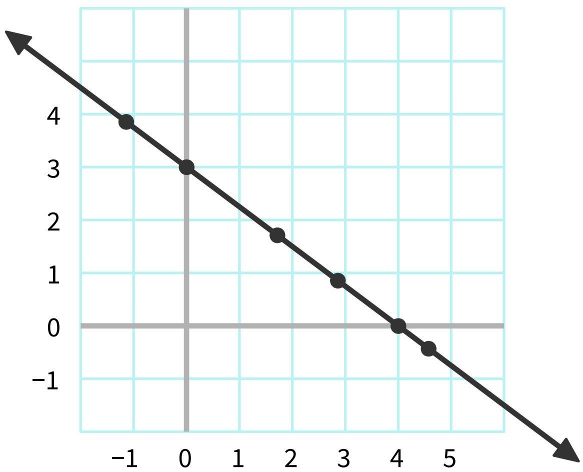
Linear Coordinate Geometry:
Straight Lines on Rectangular Planes
(Plane Lines)

1.1 Equation of a Line

We consider rectangular geometric coordinates that are *regular* (with equal unit distances in the horizontal x direction, equal to the unit distances in the vertical y direction):



In this rectangular coordinate geometry, a line is a straight direction connecting all points that are exactly in that straight direction, with both ends of the line extending infinitely. The points on the line are said to be *collinear*.



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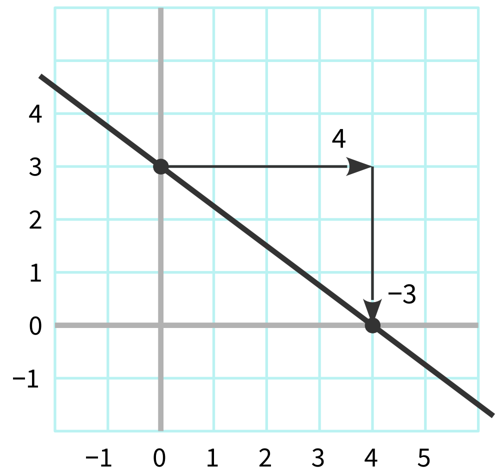
1.2 Slope

The *slope* of a line is the *rise* over *run* of the line.

For example, a line that advances 3 units vertically, for every 4 horizontal units advanced, has a slope of 3/4 or 0.75.

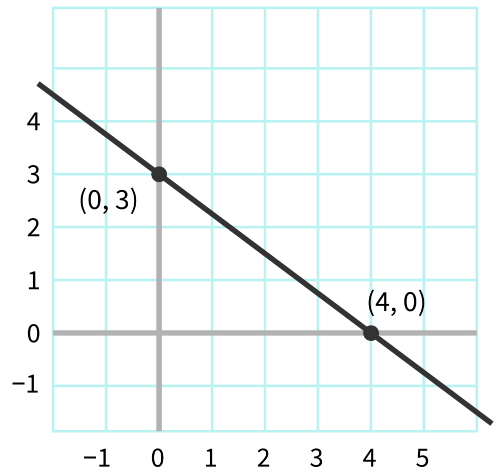
Likewise, a line that advances −3 units vertically (drops 3 units), for every 4 horizontal units advanced, has a slope of −3/4 or −0.75:



1.3 Intercepts

The *y-intercept* of a line specifies where the line intercepts the y-axis. For example, if the y-intercept is 3, the line intercepts the y-axis at (0, 3).

Likewise, the *x-intercept* specifies where the line intercepts the x-axis. For example, if the x-intercept is 4, the line intercepts the x-axis at (4, 0):



1.4 Equation of a Line

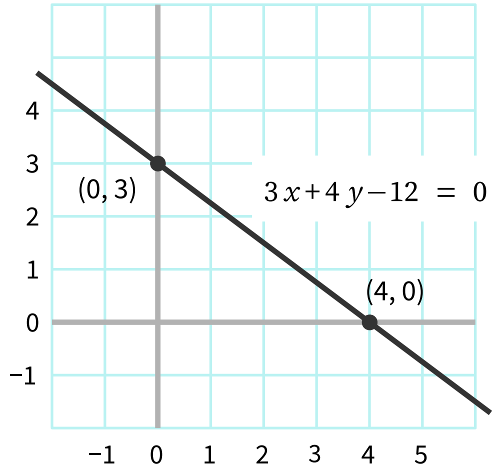
A line can be identified with an equation that specifies the location and orientation of the line. There are various equations of a line, all derivable from each other.

1.5 General Form

The general (standard) form of the equation of a line is:

$$Ax + By + C = 0$$

where A, B and C are coefficients. The slope of the line is $-(A/B)$, the x-intercept is $-(C/A)$, and the y-intercept is $-(C/B)$. (Note: the coefficients could be lower case (a, b, c) instead of upper case; we will use upper case.)



A variation of the general form of the equation of a line is:

$$Ax + By = C$$

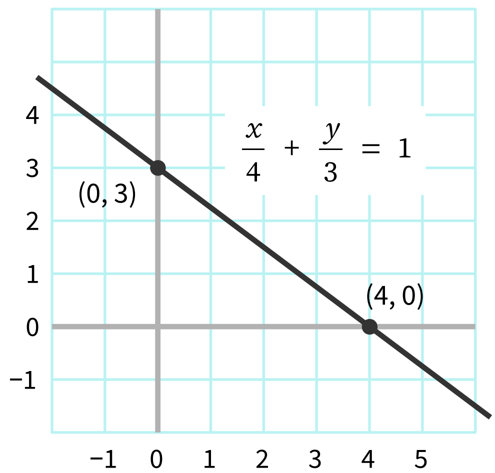
In that case, the slope of the line would be $-(A/B)$ as before, but with y-intercept C/B instead of $-C/B$.

1.6 Intercept Form

The intercept form of the equation of a line is:

$$\frac{x}{a} + \frac{y}{b} = 1$$

where a and b are the x and y intercepts respectively:

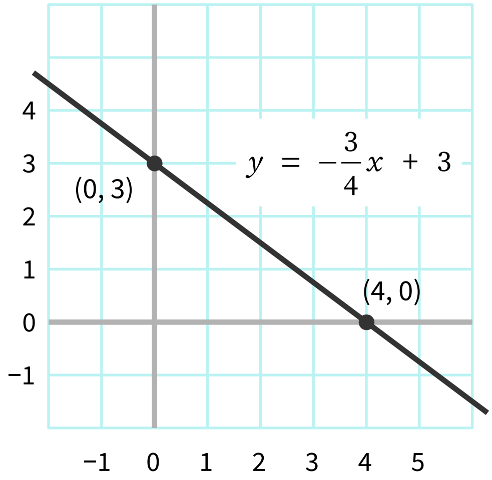


1.7 Slope Y-Intercept Form

The slope y-intercept (or simply slope-intercept) form of the equation of a line is:

$$y = mx + b$$

where m is the slope, and b is the y-intercept.



Note: In some countries, slope and y-intercept are denoted k and m respectively:

$$y = kx + m$$

We use $y = mx + b$ in this article.

1.8 Point-Slope Form

For a line with slope m, and point $P_1 = (x_1, y_1)$ on the line, the point-slope form of the equation of the line is:

$$y - y_1 = m(x - x_1)$$

1.9 Two Point Form

For a line with two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ on the line, the two point form of the equation of the line is:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

1.10 Perpendicular Slope

A useful property of the slope of a line is that the negative reciprocal of the slope will be the slope of a line perpendicular to the line: a line with slope m is perpendicular to a line with slope $-1/m$.

1.11 Intersection of Two Lines

To find the intersection point of two nonparallel lines, we can substitute the equation of one of the lines into the equation of the other line.

Given two lines:

$$A_1 x + B_1 y + C_1 = 0$$
$$A_2 x + B_2 y + C_2 = 0$$

The lines are parallel if $A_1B_2 = A_2B_1$ and are perpendicular if $A_1A_2 = -B_1B_2$.

If the lines are parallel there is no intersection. If either line is horizontal or vertical, the constant coordinate is a coordinate of the intersection, which can be substituted into the other line equation to find the other coordinate.

For example, if one of the lines is the vertical line $x = 5$, then 5 is the x-coordinate of the intersection, and substituting 5 for x in the other line equation produces the y value of the intersection.

Continuing with nonparallel lines that are non-vertical and non-horizontal, we can convert the line equations from general form to slope-intercept form.

Denoting slope and y-intercept as m and b respectively:

$$m_1 = -\frac{A_1}{B_1} \qquad m_2 = -\frac{A_2}{B_2}$$
$$b_1 = -\frac{C_1}{B_1} \qquad b_2 = -\frac{C_2}{B_2}$$

The slope-intercept form of the line equations are:

$$y = m_1 x + b_1$$
$$y = m_2 x + b_2$$

Since the y-coordinate of the intersection point will be the same for both lines, substitute one of the equations for y in the other equation:

$$m_1 x + b_1 = m_2 x + b_2$$

Solving for x:

$$m_1 x - m_2 x = b_2 - b_1$$
$$x(m_1 - m_2) = b_2 - b_1$$
$$x = (b_2 - b_1)/(m_1 - m_2)$$

The x-coordinate can then be substituted into either line equation to find the y-coordinate of the intersection.

$$y = m_1 x + b_1$$

$$y = m_1 \frac{(b_2 - b_1)}{(m_1 - m_2)} + b_1$$

To convert back to the general equation of a line ($Ax + By + C = 0$), substitute $-A/B$ for m, and $-C/B$ for b:

$$x = \frac{(b_1 - b_2)}{B_1 B_2 (A_2 B_1 - A_1 B_2)}$$

$$
\begin{aligned}
x &= \frac{(-C_2/B_2+C_1/B_1)}{(-A_1/B_1+A_2/B_2)} \\
x &= \frac{(C_1/B_1-C_2/B_2)}{(A_2/B_2-A_1/B_1)} \\
x &= \frac{B_1B_2(C_1B_2-C_2B_1)}{B_1B_2(A_2B_1-A_1B_2)} \\
x &= \frac{(C_1B_2-C_2B_1)}{(A_2B_1-A_1B_2)} \\
x &= \frac{(B_2C_1-B_1C_2)}{(A_2B_1-A_1B_2)}
\end{aligned}
$$

and

$$
\begin{aligned}
y &= m_1\frac{(b_2-b_1)}{(m_1-m_2)} + b_1 \\
y &= (-A_1/B_1)\frac{(-C_2/B_2+C_1/B_1)}{(-A_1/B_1+A_2/B_2)} - \frac{C_1}{B_1} \\
y &= -\frac{A_1(C_1/B_1-C_2/B_2)}{B_1(A_2/B_2-A_1/B_1)} - \frac{C_1}{B_1} \\
y &= -\frac{A_1(C_1/B_1-C_2/B_2)}{B_1(A_2/B_2-A_1/B_1)}\frac{B_1B_2}{B_1B_2} - \frac{C_1}{B_1} \\
y &= -\frac{A_1(B_2C_1-B_1C_2)}{B_1(A_2B_1-A_1B_2)} - \frac{C_1}{B_1}
\end{aligned}
$$

1.12 Exercise

Find the intersection of two nonparallel lines that are not vertical or horizontal, without converting to the slope-intercept form of the line equation.

Solution: As before, the y-coordinate of the intersection will be the same for both equations, so we substitute one equation for the y-coordinate in the other equation:

$$
\begin{aligned}
A_2x+B_2y+C_2 &= 0 \\
B_2y &= -(A_2x+C_2) \\
y &= -(A_2x+C_2)/B_2 \\
-(A_1x+C_1)/B_1 &= -(A_2x+C_2)/B_2 \\
(A_1x+C_1)/B_1 &= (A_2x+C_2)/B_2 \\
B_2(A_1x+C_1) &= B_1(A_2x+C_2) \\
A_1B_2x+B_2C_1 &= A_2B_1x+B_1C_2 \\
A_1B_2x-A_2B_1x &= B_1C_2-B_2C_1 \\
x(A_1B_2-A_2B_1) &= B_1C_2-B_2C_1 \\
x &= (B_1C_2-B_2C_1)/(A_1B_2-A_2B_1)
\end{aligned}
$$

Multiplying by −1/−1 gives the same result as above:

$$x = \frac{(B_2 C_1 - B_1 C_2)}{(A_2 B_1 - A_1 B_2)}$$

To solve for y, substitute x into either line equation:

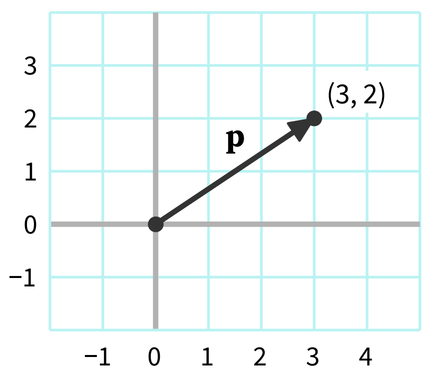
$$\begin{aligned}
A_1 x + B_1 y + C_1 &= 0 \\
B_1 y &= -(A_1 x + C_1) \\
y &= -(A_1 x + C_1)/B_1 \\
y &= -A_1 x / B_1 - C_1 / B_1 \\
y &= -\frac{A_1}{B_1} \frac{(B_2 C_1 - B_1 C_2)}{(A_2 B_1 - A_1 B_2)} - \frac{C_1}{B_1}
\end{aligned}$$

providing the same equation as above.

2.1 Vectors

The equation of a line can also be specified as a combination of vectors.

In regular coordinate geometry, a vector is a linear *distance* and *direction* specified by axis-parallel displacements. This can be used to specify the location of a point relative to another point (the other point could be a coordinate system origin). This type of vector is called a *position vector*. The following graph shows a position vector **p** that specifies the position of the point (3, 2) relative to the coordinate system origin:

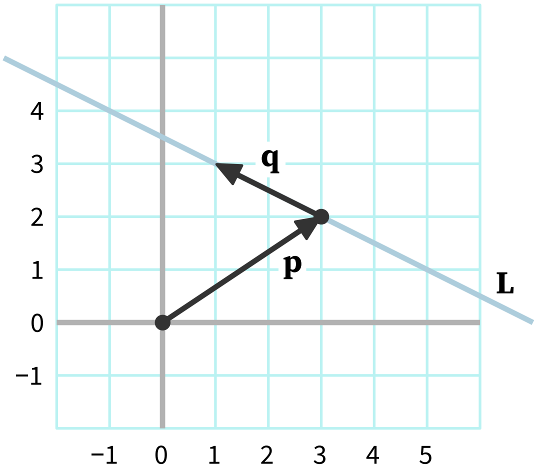


A vector can also be “free floating” to specify the direction of flow in a field, referred to as a *free vector*.

Some vectors may have a predefined fixed length, in order to only specify direction, referred to as a *direction vector*. If the fixed length of the direction vector is equal to one (1), it is a *unit vector*.

A direction vector may be a position vector or a free vector. If the direction vector is a position vector, it is collinear with the line that has the same direction and passes through that position vector. If the direction vector is a free vector, it is collinear with all parallel lines that have that direction.

Consider a position vector **p** as described above to specify a point on a line, and a vector **q** that points in the same direction as the line. We can position the base of **q** at the end (tip) of **p** to specify a line that passes through **p** and has direction of **q**.



This vector equation of a line is:

$$\mathbf{L} = \mathbf{p} + t\mathbf{q}$$

where **L** is the position vector to any point on the line, **p** is the position vector to a predefined point on the line, **q** is the direction vector that has the same direction as the line, and t is a scalar that is applied to **q** (performing scalar multiplication of **q**) to find a point on the line.

If the parameter t is zero, the point on the line is at the predefined point on the line (at the end of **p**). If t is positive, the point on the line is in the **q** direction from the predefined point. If t is negative, the point is in the opposite direction.

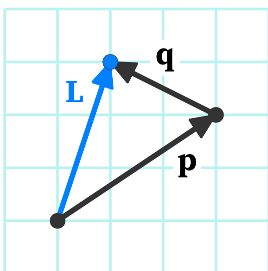
2.2 Vector Addition

Vector addition consists of adding corresponding components of vectors:

$$(x_0,\ y_0) + (x_1,\ y_1) = (x_0 + x_1,\ y_0 + y_1)$$

Only the corresponding components are added together. Components are not mixed — that would be like adding “apples and oranges” so to speak.

For **p** = (3, 2) and **q** = (−2, 1), **L** = **p** + **q** = (3−2, 2+1) = (1, 3):

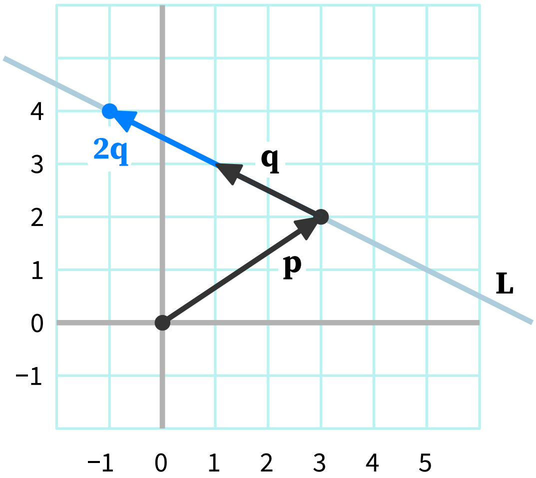


2.3 Scalar Multiplication

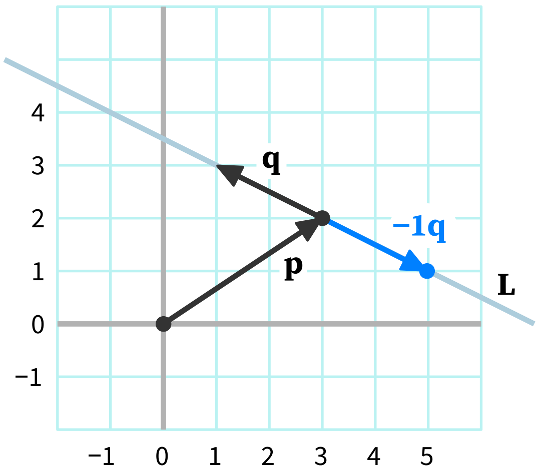
A scalar is a single number. Scalar multiplication of a vector multiplies each vector component with the scalar (which scales the vector). For a scalar t:

$$t\,(x,\ y) = (tx,\ ty)$$

For $t = 2$:



For $t = -1$:



2.4 Scalar Product

Unlike scalar multiplication, which operates on a single vector (multiplying the vector with a scalar), vector algebra also has *products*. The “products” of vector algebra operate on two vectors at a time, producing a scalar or a vector.

The type of product that produces a scalar is called a *scalar product*. That is defined in this section. The type of product that produces a vector is called a *vector product*, and is defined in the next section.

A scalar is a single number (not a vector). A scalar product (also called *inner product* or *dot product*) operates on two vectors to produce a scalar, using the following formula for the rectangular coordinates we are using:

$$(x_0,\ y_0) \cdot (x_1,\ y_1) = x_0 x_1 + y_0 y_1$$

with a dot (·) as the operator symbol for producing a scalar product.

A useful feature of the scalar product is to calculate the length of a projection of a vector on another vector (explained below).

2.5 Vector Product

A vector product (also called *cross product*) operates on two vectors to produce a vector, using the following formula for two vectors on the *xy* plane:

$$(x_0,\ y_0,\ 0) \times (x_1,\ y_1,\ 0) = (0,\ 0,\ x_0 y_1 - x_1 y_0)$$

with a crossing symbol (×) as the operator symbol for producing a vector product.

Vector products actually operate in three dimensional space (*x*, *y*, *z*), producing a vector that is perpendicular to the two vectors being crossed. In our case, we are setting the third coordinate (*z*) of the crossed vectors to zero, making them lie on the *xy* plane, producing a vector that is entirely in the *z* direction (not on the *xy* plane).

The length of the vector that is produced (in this case its *z* value) provides the area of the parallelogram enclosed by the crossed vectors on the *xy* plane, allowing us to use that value as a scalar (since this cross product does not have *x* or *y* values). The sign of that number is positive above the *xy* plane, or negative below the *xy* plane. Which side is above or below depends on the rotation direction from the first crossed vector to the second crossed vector, and for some applications is not important other than to compare to other vectors.

2.6 Vector Length

Vector length is a scalar, that is the distance, from the base of the vector to its tip (from the the tail to the head of the vector). Vector length is denoted with vertical bars (absolute value). It is calculated with the Pythagorean Theorem. For a two-dimensional vector **u** = (u_x, u_y):

$$|\mathbf{u}| = \sqrt{u_x u_x + u_y u_y}$$

and in three dimensions, for **u** = (u_x, u_y, u_z):

$$|\mathbf{u}| = \sqrt{u_x u_x + u_y u_y + u_z u_z}$$

Note that if a vector has only one nonzero component, the absolute value of that component is the vector length:

$$|(0,4)| = \sqrt{(0)(0)+(4)(4)} = 4$$
$$|(-4,0)| = \sqrt{(-4)(-4)+(0)(0)} = 4$$

$$|(0,0,5)| = \sqrt{(0)(0)+(0)(0)+(5)(5)} = 5$$
$$|(0,0,-5)| = \sqrt{(0)(0)+(0)(0)+(-5)(-5)} = 5$$

That is why we can use the *z* component of a cross product of two *xy* vectors as the length of the cross product. Since the cross product is perpendicular to both crossed vectors, none of the cross product components are on the plane of the crossed vectors.

2.7 Unit Vector

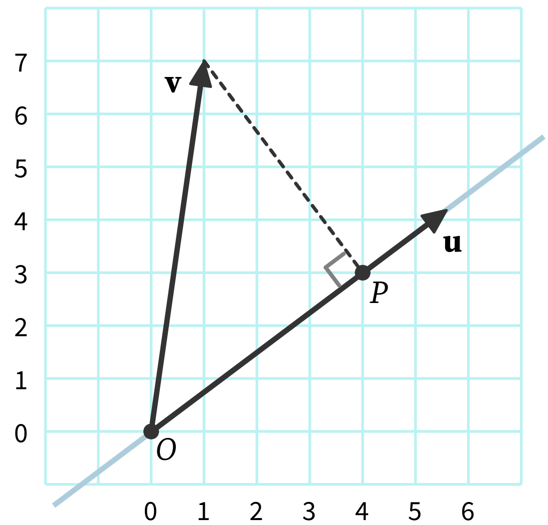
A unit vector is a vector with length equal to one (1), sometimes denoted with a "hat" (^). To convert a vector to a unit vector with the same direction, scale the vector by the reciprocal of its length. For a vector $\mathbf{u} = (u_x, u_y)$:

$$\text{unit vector of } \mathbf{u} = \hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|} = \left(\frac{u_x}{|\mathbf{u}|}, \frac{u_y}{|\mathbf{u}|}\right)$$

3.1 Projection

A useful feature of the scalar product is to calculate the length of a projection of a vector on the line of another vector.

For two vectors $\mathbf{u} = (5.6, 4.2)$ and $\mathbf{v} = (1, 7)$, with common origin denoted O, the perpendicular projection of $\mathbf{v}$ on the line containing $\mathbf{u}$ is a point denoted P. The point P is called the perpendicular foot of $\mathbf{v}$ on the line of $\mathbf{u}$.



The distance from O to P (denoted OP) is proportional to the scalar product of $\mathbf{u}$ and $\mathbf{v}$. The distance OP is the scalar product $\mathbf{u} \cdot \mathbf{v}$ multiplied by the length of $\mathbf{u}$. To obtain the distance OP, divide the scalar product $\mathbf{u} \cdot \mathbf{v}$ by the length of $\mathbf{u}$.

The length of $\mathbf{u}$ is 7:

$$\begin{aligned}
\mathbf{u} &= (5.6, 4.2) \\
|\mathbf{u}| &= |(5.6, 4.2)| \\
&= \sqrt{5.6^2 + 4.2^2} \\
&= \sqrt{31.36 + 17.64} \\
&= \sqrt{49} \\
&= 7
\end{aligned}$$

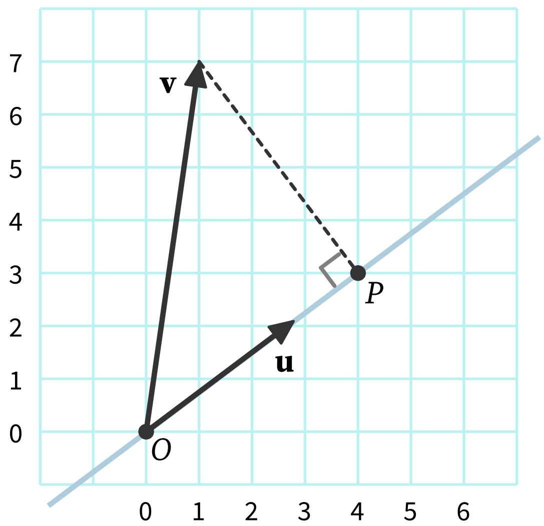
The scalar product $\mathbf{u} \cdot \mathbf{v}$ equals 35:

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= (5.6, 4.2) \cdot (1, 7) \\
&= (5.6)(1) + (4.2)(7) \\
&= 5.6 + 29.4 \\
&= 35
\end{aligned}$$

Dividing **u** · **v** = 35 with |**u**| = 7 yields the distance *OP* = 5. That is the hypotenuse of a 3/4/5 triangle. Note that division by |**u**| would not have been necessary if **u** was a unit vector (because a unit vector length is 1).

Consider **u** = (2.8, 2.1). Then the length of **u** is 3.5:

$$\begin{aligned}\mathbf{u} &= (2.8\,,2.1)\\ |\mathbf{u}| &= |(2.8\,,2.1)|\\ &= \sqrt{2.8^2+2.1^2}\\ &= \sqrt{7.84+4.41}\\ &= \sqrt{12.25}\\ &= 3.5\end{aligned}$$

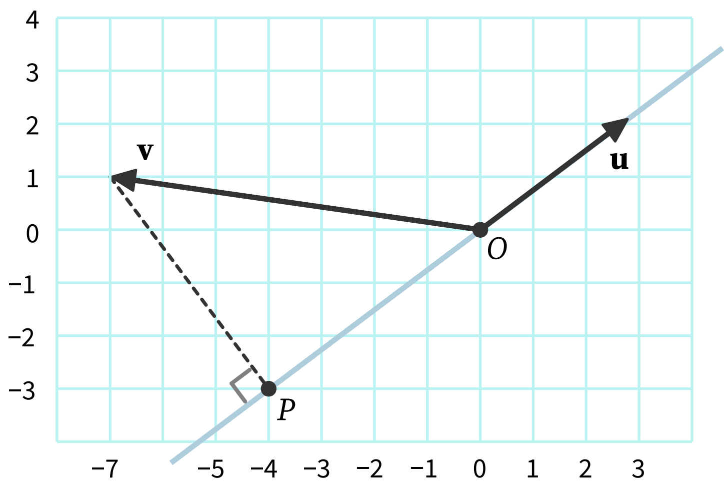


The scalar product **u** · **v** equals 17.5:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= (2.8\,,2.1)\cdot(1\,,7)\\ &= (2.8)(1)+(2.1)(7)\\ &= 2.8+14.7\\ &= 17.5\end{aligned}$$

Dividing **u** · **v** = 17.5 with |**u**| = 3.5 yields the distance *OP* = 5 again.

In the preceding examples, the angle subtending (between) **u** and **v** was *acute* (less than 90°). Now consider **v** = (−7, 1) which subtends an *obtuse* angle with **u** (greater than 90°):



The scalar product $\mathbf{u} \cdot \mathbf{v}$ equals −17.5:

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= (2.8\,,2.1)\ \cdot\ (-7\,,1) \\ &= (2.8)(-7)+(2.1)(1) \\ &= -19.6+2.1 \\ &= -17.5 \end{aligned}$$

Dividing $\mathbf{u} \cdot \mathbf{v} = -17.5$ with $|\mathbf{u}| = 3.5$ yields the distance $OP = -5$. The distance is negative because it is in the opposite direction than **u** on the line containing **u**.

3.2 Perpendicular Foot Point

The perpendicular projection of **v** on the line of **u** is the point denoted P. The point P is called the perpendicular foot of **v** on the line of **u**. The coordinates of P can be found by scaling the unit vector of **u** with the distance OP.

$$\text{unit vector of } \mathbf{u} = \hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|} = \left(\frac{u_x}{|\mathbf{u}|}, \frac{u_y}{|\mathbf{u}|}\right)$$

$$\hat{\mathbf{u}} = \left(\frac{2.8}{3.5}, \frac{2.1}{3.5}\right) = (0.8\,,0.6)$$

$$\begin{aligned} OP\,\hat{\mathbf{u}} &= -5(0.8\,,0.6) \\ &= (-4\,,-3) \end{aligned}$$

3.3 Vector Subtraction (Distance to Line)

Vector subtraction is simply vector addition with one of the vectors scaled with −1. The result is a vector that extends from the head of one of the vectors to the head of the other vector. In this case, it gives us a vector joining the head of **v** with the point P (head of $OP\,\hat{\mathbf{u}}$). The length of that vector is the shortest distance from the head of **v** to the line containing P.

Denoting that vector as **d**:

$$\begin{aligned} \mathbf{d} &= \mathbf{v} - OP\,\hat{\mathbf{u}} \\ &= \mathbf{v} + (-OP\,\hat{\mathbf{u}}) \\ &= (-7,1) + (4,3) \\ &= (-3,4) \end{aligned}$$

The length of **d** is the shortest distance from the head of **v** to the line of **u**.

$$\begin{aligned} |\mathbf{d}| &= \sqrt{-3^2+4^2} \\ &= \sqrt{9+16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

Sometimes we only need the square of the distance, to compare to other squared distances — in that case, the square root may be skipped.

4.1 Cross Products

The scalar (dot) product changes sign when the angle subtending the dotted vectors changes from acute (less than 90°) to obtuse (greater than 90°), as shown in the previous examples. However, dot product does not change sign depending on which side of the line the other vector is — you can verify that by swapping the order of the vectors, producing the same result.

The cross product does change sign when the other vector is on the other side. It does not change sign when the subtending angle becomes obtuse instead of acute, but it does change sign when the second vector is on the other side of the line.

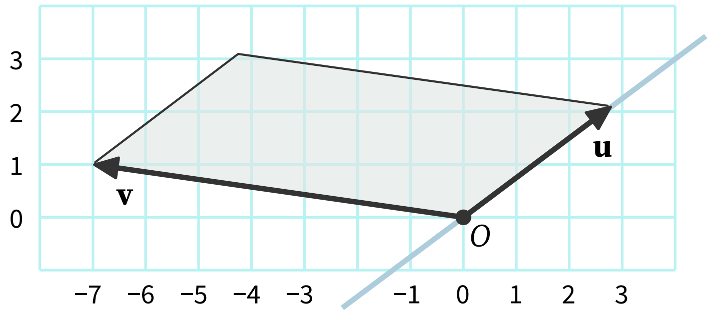
For finding out if a vector is obtuse, use the dot product. For finding out which side a vector is located, use the cross product.

4.2 Parallelogram

The portion of a plane that is on one side of a line is called a *half plane*. The two sides of a line, and the line itself, together make up an entire plane.

A parallelogram is a quadrilateral with opposite parallel sides, with each side equal length to the side that is parallel to it. Since the opposite sides are equal, a parallelogram is convex. Two *xy* vectors with common origins form two sides of a parallelogram, which defines the parallelogram because the other two sides will be parallel and equal length (and therefore subtend the first two sides).

Continuing with the vectors **u** = (2.8, 2.1) and **v** = (−7, 1) of the preceding example, the parallelogram of those vectors is illustrated as follows:



The area of a parallelogram is *base* times *height*. In this example, the base and height can be the vector length of **u** and the distance |**d**| from the head of **v** to the line of **u** (both of which were calculated in the preceding example):

$$\begin{aligned} \text{Area} &= |\mathbf{d}||\mathbf{u}| \\ &= (5)(3.5) \\ &= 17.5 \end{aligned}$$

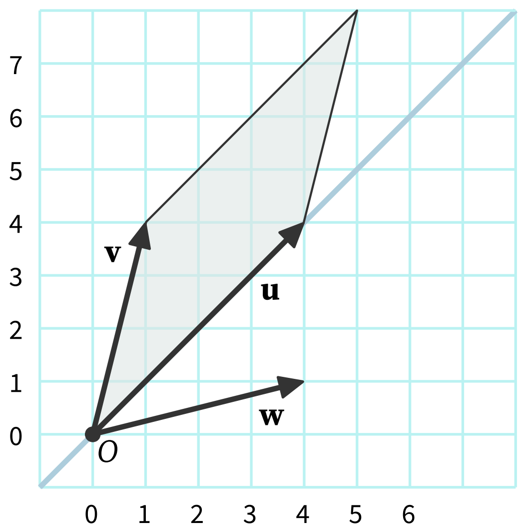
The area of a parallelogram may also be calculated as a cross product subtending two adjacent sides. The cross product of the two *xy* vectors provides a *z* value that is the area of the parallelogram defined by the *xy* vectors.

For example, the cross product **u** × **v** of **u** = (2.8, 2.1) and **v** = (−7, 1), in its *z* component, gives the area of the parallelogram bounded by **u** and **v**:

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_x, u_y) \times (v_x, v_y) \\ &= (0, 0, u_x v_y - v_x u_y) \\ &= (0, 0, (2.8)1 - (-7)2.1) \\ &= (0, 0, 2.8 + 14.7) \\ &= (0, 0, 17.5) \end{aligned}$$

as calculated above (this time using the cross product).

Now consider three vectors: **u** = (4, 4); **v** = (1, 4); and **w** = (4, 1). The following graph shows the parallelogram bounded by **u** and **v**:



The cross product $\mathbf{u} \times \mathbf{v}$ calculates that the area of the parallelogram is 12:

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_x,\ u_y) \times (v_x,\ v_y) \\ &= (0,\ 0,\ u_x v_y - v_x u_y) \\ &= (0,\ 0,\ (4)4 - (1)4) \\ &= (0,\ 0,\ 16 - 4) \\ &= (0,\ 0,\ 12)\end{aligned}$$

The vector $\mathbf{w}$ is the mirror of $\mathbf{v}$ on the other side of $\mathbf{u}$. Since it is a mirror image on the other half plane, it should have the same area. Using the cross product to calculate the area of the parallelogram bounded by $\mathbf{u}$ and w:

$$\begin{aligned}\mathbf{u} \times \mathbf{w} &= (0,\ 0,\ u_x w_y - w_x u_y) \\ &= (0,\ 0,\ (1)4 - (4)4) \\ &= (0,\ 0,\ 4 - 16) \\ &= (0,\ 0,\ -12)\end{aligned}$$

The area is the same, but negative instead of positive, indicating it is in the other half plane.

This property of changing sign when changing half plane is useful for determining if an arbitrary point is inside of a triangle on a plane.

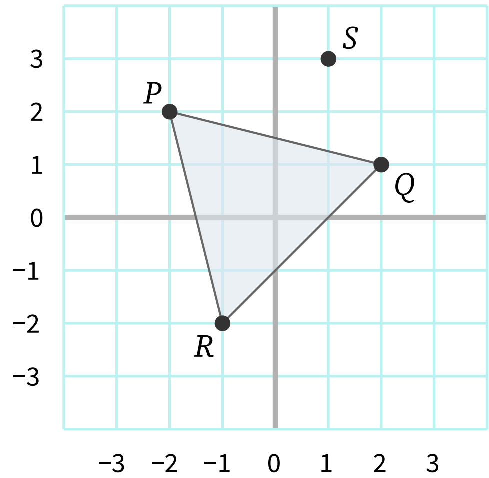
4.3 Point in Triangle

For a given point on a plane that contains a triangle, to find out if the point is inside the triangle, we can calculate the cross product of the point relative to each of the three sides of the triangle (three cross products) and compare the signs.

The traversal sequence must be the same for all three sides: clockwise, or counter-clockwise. Taking a triangle vertex as the origin (common tail) for two vectors, the first vector ends (with head) at the next triangle vertex in the traversal sequence, and the second vector ends at the given point which may or may not be in the triangle.

If all the cross products have the same sign, the point is in the triangle. If one of the cross products is zero, the point is on an edge of the triangle. If the cross products are nonzero and not all the same sign, the point is not in the triangle.

Consider whether the point $S = (1, 3)$ is in the triangle with vertices $P = (-2, 2)$, $Q = (2, 1)$ and $R = (-1, -2)$:



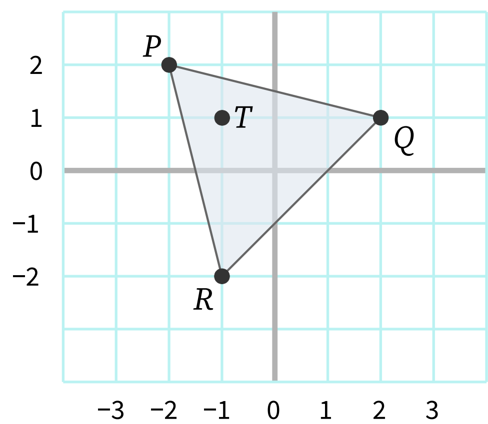
In clockwise orientation, the cross products of the sides and the point are:

$$
\begin{aligned}
PQ \times PS &= (4,\ -1,\ 0) \times (3,\ 1,\ 0) \\
&= (0,\ 0,\ 4(1) - 3(-1)) \\
&= (0,\ 0,\ 4 + 3) \\
&= (0,\ 0,\ 7) \\
QR \times QS &= (-3,\ -3,\ 0) \times (-1,\ 2,\ 0) \\
&= (0,\ 0,\ -3(2) - (-1)(-3)) \\
&= (0,\ 0,\ -6 - 3) \\
&= (0,\ 0,\ -9) \\
RP \times RS &= (-1,\ 4,\ 0) \times (2,\ 5,\ 0) \\
&= (0,\ 0,\ -1(5) - 2(4)) \\
&= (0,\ 0,\ -5 - 8) \\
&= (0,\ 0,\ -13)
\end{aligned}
$$

The signs are not the same, indicating the point is outside of the triangle.

Try counter-clockwise orientation to find that the signs are different but still not the same.

Now consider whether the point $T = (-1, 1)$ is in the triangle PQR:



The cross products in clockwise orientation are:

$$\begin{aligned}
PQ \times PT &= (4, -1, 0) \times (1, -1, 0) \\
&= (0, 0, 4(-1) - 1(-1)) \\
&= (0, 0, -4 + 1) \\
&= (0, 0, -3) \\
QR \times QT &= (-3, -3, 0) \times (-3, 0, 0) \\
&= (0, 0, -3(0) - (-3)(-3)) \\
&= (0, 0, 0 - 9) \\
&= (0, 0, -9) \\
RP \times RT &= (-1, 4, 0) \times (0, 3, 0) \\
&= (0, 0, -1(3) - 0(4)) \\
&= (0, 0, -3 - 0) \\
&= (0, 0, -1)
\end{aligned}$$

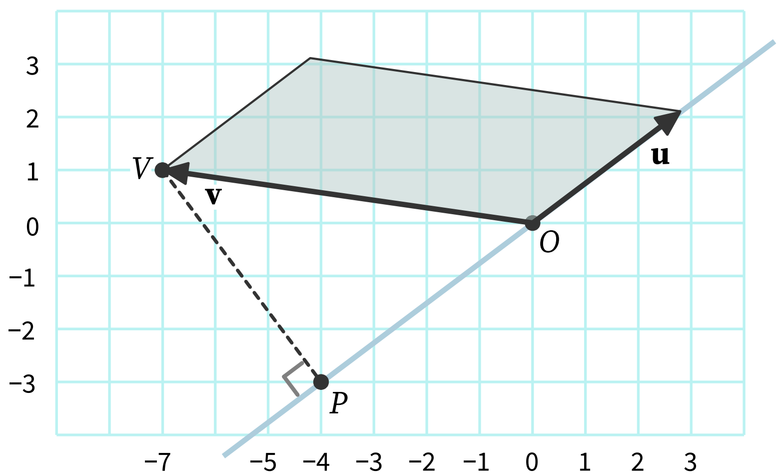
The signs are the same, indicating the point is in the triangle.

Try calculating the cross products in counter-clockwise orientation to find that the signs are all positive (a different sign, but still all the same signs indicating the point is in the triangle).

Note: If one of the cross products is zero, the point is on that edge of the triangle. For some applications, that may be considered to be in the triangle (depending on the application).

4.4 Distance to Line

The distance from a point to a line is the distance from the point to its perpendicular foot on the line (Section 3.2 above). Returning to the example of vectors $\mathbf{u} = (2.8, 2.1)$ and $\mathbf{v} = (-7, 1)$, the problem is to find the distance from the tip of $\mathbf{v}$ to its perpendicular foot P on the line of $\mathbf{u}$ this time using a cross product. Denoting the tip of $\mathbf{v}$ as V:



The cross product $\mathbf{u} \times \mathbf{v}$ calculates the area of the parallelogram:

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_x,\ u_y) \times (v_x,\ v_y) \\
&= (0,\ 0,\ u_x v_y - v_x u_y) \\
&= (0,\ 0,\ (2.8)1 - (-7)2.1) \\
&= (0,\ 0,\ 2.8 + 14.7) \\
&= (0,\ 0,\ 17.5)
\end{aligned}$$

The area of the parallelogram bounded by **u** and **v** is 17.5.

From plane geometry, the area of a parallelogram is *base* times *height*. The base of this parallelogram is the length of **u**:

$$\begin{aligned}
\mathbf{u} &= (2.8, 2.1) \\
|\mathbf{u}| &= |(2.8, 2.1)| \\
&= \sqrt{2.8^2 + 2.1^2} \\
&= \sqrt{7.84 + 4.41} \\
&= \sqrt{12.25} \\
&= 3.5
\end{aligned}$$

The height of the parallelogram is the distance we are looking for (the distance from P to V). Dividing through the *Area = base times height* formula with *base*:

$$\begin{aligned}
\text{height of parallelogram} &= \text{Area / base} \\
PV &= 17.5 / 3.5 \\
&= 5
\end{aligned}$$

The distance from point V to the line of **u** is 5, as calculated in Section 3.3 above, this time calculated using a cross product.